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Enrichment Programme for Young Mathematics Talents

二零二四至二五年度學期科目考試

Course Final Examination, 2024-25

科目編號及名稱 Course Code & Title : **SAYT1134 Towards Differential Geometry**

時間 Time allowed : **2** 小時 hours **30** 分鐘 minutes

姓名 Name : ____________ 年級 Form : ____________

Instructions

- The full mark of the paper is **100 points** and bonus mark **20 points**.
- This paper consists of **Basic Part**, **Harder Part** and **Bonus Part**.
- Answer **ALL** questions in Basic Part and **FIVE** questions in Harder Part. Make your best effort to answer the Bonus Part.
- Show your work clearly and neatly. Give adequate explanation and justification for your calculation and observation.
- Write your answers in the spaces provided in the Answer Booklet. Begin each question on a new page. Clearly indicate the question number in the designated slot at the top of each page.
- Supplementary answer sheets and rough paper will be supplied on request.
- Non-graphical calculators are allowed.

Useful Formulaes:

- Trigonometry

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

- Curve curvature
  - $\kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \left\langle \dfrac{\mathbf{T}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{N}(t) \right\rangle$
  - For space curve: $\kappa(t) = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$
- Space curve torsion
  - $\tau(t) = \left\langle \dfrac{\mathbf{N}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{B}(t) \right\rangle$
  - $\tau(t) = \dfrac{\langle \mathbf{r}'(t) \times \mathbf{r}''(t), \mathbf{r}'''(t) \rangle}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$
- Frenet formula: $\dfrac{d}{ds}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$
- Gaussian Curvature
  - Let S be a regular surface parametrized by $\mathbf{x}(u, v)$ and K be the Gaussian curvature of S. Then
    $$K = \det(d\mathbf{n}_p) = \frac{\det(II)}{\det(I)} = \frac{eg - f^2}{EG - F^2}$$
    where I is the first fundamental form and II is the second fundamental form of the surface.
  - For isothermal parametrization on surface with first fundamental form $I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$, the Gaussian curvature is given by
    $$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$
- The matrix representation of differential of Gauss map $d\mathbf{n}_p$ with respect to basis $\mathbf{x}_u, \mathbf{x}_v$ is
  $$d\mathbf{n}_p = -(II)(I)^{-1} = -\frac{1}{EG - F^2}\begin{pmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{pmatrix}$$
- Let S be a regular surface and $d\mathbf{n}_p$ be the differential of Gauss map at $p \in S$. The mean curvature of S at p is
  $$H = -\frac{1}{2}\mathrm{tr}(d\mathbf{n}_p) = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right)$$

Basic Part (50 points). Answer **ALL** questions in this part.

1. (a) (3 points) Which of the following is/are regular curves? Provide a short explanation.
   (i) $\gamma_1 : \mathbb{R} \to \mathbb{R}^2$, $\gamma_1(t) = (t^3, t^2 + 1)$; and
   (ii) $\gamma_2 : \mathbb{R} \to \mathbb{R}^3$, $\gamma_2(t) = (t^2, 1 - t^2, 1 + t^3)$.

   (b) (7 points) Given a regular space curve $\mathbf{r} : (a, b) \to \mathbb{R}^3$ parametrized by arc length with positive curvature $\kappa(s) > 0$, for all $s \in (a, b)$.
   (i) Show that if $\mathbf{T} + \mathbf{B}$ is constant for all $s \in (a, b)$, then
   $$\kappa(s) = \tau(s)$$
   for all $s \in (a, b)$.
   (ii) Simplify the expression $(\mathbf{T}' \times \mathbf{N}') \cdot \mathbf{B}$.

2. (5 points) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a parametrized space curve defined by
   $$\gamma(t) = \left(3t - t^3, 3t^2, 3t + t^3\right), \ t \in \mathbb{R}.$$
   Show that
   $$\kappa(t) = \tau(t) = \frac{1}{3(1 + t^2)^2}.$$

3. Let S be the surface parametrized by
   $$X(u, v) = (uv^2, u - v, u + v), \quad (u, v) \in \mathbb{R}^2$$
   and X is a smooth map.
   (a) (5 points) Compute $X_u \times X_v$. Hence, show that S is a regular surface.
   (b) (5 points) Compute the first fundamental form of X as a 2×2 matrix and the second fundamental form of X as a 2×2 matrix with respect to the unit normal vector $N = X_u \times X_v / \|X_u \times X_v\|$.
   (c) (5 points) Consider the point $p = (0, 1, 1) \in S$, compute the Gaussian curvature K and the mean curvature H of S at p.

4. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the inverse of the *stereographic projection* defined by
   $$\Phi(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2}\right)$$
   where $(u, v) \in \mathbb{R}^2$. Suppose that $B = \Phi(\mathbb{R}^2)$ is a surface.
   (a) (5 points) Compute $\Phi_u \times \Phi_v$. Hence, show that Φ is regular.
   (b) (5 points) Compute the first fundamental form of Φ as a 2×2 matrix.
   (c) (5 points) Hence, compute the Gaussian curvature K of B .
   (d) (5 points) Clive claims that $\Phi(u, v)$ is a parametrization for whole unit sphere centered at the origin. Do you agree? Explain your answer.

Harder Part (10+40 points).

Answer **ALL** True/False Questions. Answer **FOUR** questions in **Structured Questions**.

True/False Questions

5. (10 points) Mark each of the following statements "True" (meaning that it is a true statement) or "False" (meaning that there are counterexamples to the statement). **Brief justification is required**. Each correct answer with correct justification carries 2 points, correct answer without correct justification carries 1 point and wrong answer with/without justification carries 0 point.
(**Note.** *You may use any results in lecture notes or tutorial notes.*)
   (a) For any given smooth functions $\kappa(s), \tau(s) > 0$, there exists unique regular curve $\mathbf{r}(s)$ in $\mathbb{R}^3$ parametrized by arc length with curvature $\kappa(s)$ and torsion $\tau(s)$.
   (b) If S is a regular surface with Gaussian curvature $K > 0$ everywhere, then the curvature of any regular curves $C \subset S$ is everywhere positive.
   (c) If the Gauss map of a regular surface S in $\mathbb{R}^3$ is constant, then S is contained in a flat plane.
   (d) Let S_1 and S_2 be two regular surfaces. If $K(p) = K(q)$ for any $p \in S_1$, $q \in S_2$, where K denotes the Gaussian curvature, then S_1 and S_2 are isometric.
   (e) There is a minimal surface S in $\mathbb{R}^3$ satisfying $K(p) > 0$ for some $p \in S$.

Structured Questions

6. (a) (5 points) A **constant-speed curve** with speed c is a space curve $\mathbf{r} : (a, b) \to \mathbb{R}^3$ with $\|\mathbf{r}'(t)\| = c$ for every $a < t < b$, where c is a positive constant. Suppose that $\mathbf{r}$ is a regular curve, show that $\mathbf{r}$ is a constant-speed curve *if and only if* $\langle \mathbf{r}', \mathbf{r}'' \rangle = 0$ for all $a < t < b$.
   (b) (5 points) Let $\mathbf{r}$ be a constant-speed curve with speed c, define the following two curves:

$$\boldsymbol{\alpha} : \left(-\frac{b}{3}, -\frac{a}{3}\right) \to \mathbb{R}^3 : \qquad \boldsymbol{\alpha}(t) := \mathbf{r}(-3t)$$
$$\boldsymbol{\beta} : (a, b) \to \mathbb{R}^3 : \qquad \boldsymbol{\beta}(t) := 5\mathbf{r}(t)$$

   Let $\kappa(t)$ be the curvature of $\mathbf{r}$ at point t. Express the curvature of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in terms of $\kappa(t)$ and c.

7. Let $\mathbb{D}_n = (0, \sqrt{n}\pi) \times (0, 2\pi)$, $n \in \mathbb{N}$ be an open connected subset in $\mathbb{R}^2$.
   (a) (5 points) Suppose $\Phi(r,\theta)$ for $(r,\theta) \in \mathbb{D}_1$ has the following first fundamental form:
   $$I = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 r \end{pmatrix}.$$
   Find the Gauss curvature K for this surface.
   (b) (5 points) Suppose $\tilde{\Phi}(r,\theta)$ for $(r,\theta) \in \mathbb{D}_{2024}$ has the following first fundamental form:
   $$\tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & 2024\sin^2\left(\frac{1}{\sqrt{2024}}r\right) \end{pmatrix}.$$
   Let $\widetilde{K}$ be the Gauss curvature for the parametrization $\widetilde{\Phi}$.
   Show that
   $$\iint_{\mathbb{D}_1} K\sqrt{\det I}\, \mathrm{d}r\mathrm{d}\theta = \iint_{\mathbb{D}_{2024}} \widetilde{K}\sqrt{\det \widetilde{I}}\, \mathrm{d}r\mathrm{d}\theta = 4\pi.$$

8. (a) (3 points) State the **Gauss-Bonnet Theorem** for smooth, closed orientable surfaces in $\mathbb{R}^3$. *(Define the symbols you have used.)*
   (b) (2 points) Suggest a surface with Gaussian curvature $K \geq 2024$ and surface area $A = \pi/506$, then verify your answer.
   (c) (5 points) Let S_1, S_2 be two closed orientable surfaces. Given that
   - S_1 has Gaussian curvature $K(p) \geq 1$ for all $p \in S_1$.
   - S_2 has constant Gaussian curvature 1.
   - S_1, S_2 have the same Euler characteristic.

   Show that $\text{Area}(S_1) \leq \text{Area}(S_2)$.

9. Let S be a regular closed orientable surface with genus $g \geq 1$.
   (*Note. By the Classification Theorem of compact connected surfaces in* $\mathbb{R}^3$, *S is not homeomorphic to the sphere.*)
   (a) (5 points) Show that K attains both positive and negative values.
   (b) (5 points) Is it necessary that S has zero Gaussian curvature at some point? Explain your answer.

10. Let $X(u,v)$ be a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^3$. It is given that the first fundamental form of X as a 2×2 matrix is

$$I = \begin{pmatrix} f(u,v) & 0 \\ 0 & f(u,v) \end{pmatrix}$$

where $f(u,v) > 0$ is a smooth function. Let $M = X(D)$ be a surface in $\mathbb{R}^3$.

(a) (5 points) Denote the Gaussian curvature of M by $K(u,v)$.
Show that

$$K(u,v) = -\frac{1}{2f}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ln f(u,v).$$

(b) (2 points) Suppose that $f(u,v) = e^{-2u^2}$.
Show that $K > 0$ for any $(u,v) \in \mathbb{R}^2$.

(c) (3 points) Is it possible to find a *closed* connected surface in $\mathbb{R}^3$ under a parametrization X such that its first fundamental form is

$$I(u,v) = \begin{pmatrix} e^{-2u^2} & 0 \\ 0 & e^{-2u^2} \end{pmatrix}$$

which is exactly the same as part (b)? Explain your answer.

Bonus Part (20 points). Try your best to answer the question in this part.

11. Recall that for a $n \times n$ matrix A with real entries, we denote $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ to be an eigenvector with eigenvalue $\lambda \in \mathbb{R}$ if we have

$$A\mathbf{x} = \lambda \mathbf{x}.$$

   (a) (2 points) If we have $A^k = \mathbf{O}$ for some $k \in \mathbb{N} \setminus \{0, 1\}$, determine the possible eigenvalue(s) that A might have.

   (b) Let A be a $n \times n$ symmetric matrix (i.e. $A^T = A$) with real entries such that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \setminus \{\mathbf{0}\}$ are eigenvectors with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ respectively. Suppose $\lambda_1 < \lambda_2 < ... < \lambda_n$ and $\|\mathbf{x}_i\| = 1$ for all $i = 1, 2, ..., n$.

      (i) (5 points) Prove that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a linearly independent and orthogonal set. Hence, $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ forms an orthonormal basis of $\mathbb{R}^n$.

      (ii) (3 points) Using (b)(i), show that for all $\mathbf{x} \in \mathbb{R}^n$, we have

$$\lambda_1 \|\mathbf{x}\|^2 \leq \langle \mathbf{x}, A\mathbf{x} \rangle \leq \lambda_n \|\mathbf{x}\|^2.$$

      (iii) (2 points) Is (b)(ii) true for general $n \times n$ matrix A with real entries? Explain your answer.

   (c) (i) (2 points) Let A be a 2×2 matrix with real entries such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we have

$$\langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle.$$

      Show that A is symmetric.

      (ii) Suppose $\mathbf{p}$ is a point on a regular surface $S \subset \mathbb{R}^3$, tangent plane $T_{\mathbf{p}}S$ with basis $\{\mathbf{X}_u, \mathbf{X}_v\}$. (As $T_{\mathbf{p}}S$ can be transformed to $\mathbb{R}^2$ through rotation and translation, you can replace $\mathbb{R}^2$ by $T_{\mathbf{p}}S$ in (b)(i) and (c)(i).)

         (1) (3 points) Let $B_{\mathbf{p}}$ be a 2×2 matrix representation of the shape operator $S_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ ($S_{\mathbf{p}}(\mathbf{v}) = -d\mathbf{n}_{\mathbf{p}}(\mathbf{v})$.) on $\mathbf{p}$ (which exists as $S_{\mathbf{p}}$ is linear). Using (c)(i), show that $B_{\mathbf{p}}$ is symmetric.

         (2) (3 points) Suppose further that
         - $S_{\mathbf{p}}(\mathbf{X}_u) = \lambda_1 \mathbf{X}_u$, $S_{\mathbf{p}}(\mathbf{X}_v) = \lambda_2 \mathbf{X}_v$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$.
         - $\|\mathbf{X}_u\| = \|\mathbf{X}_v\| = 1$.

         Using *(b)(i)* Show that for each unit vector $\mathbf{x} \in T_{\mathbf{p}}S$, there exists $\theta_{\mathbf{x}} \in \mathbb{R}$ such that

$$\langle \mathbf{x}, B_{\mathbf{p}}\mathbf{x} \rangle = \lambda_1 \cos^2 \theta_{\mathbf{x}} + \lambda_2 \sin^2 \theta_{\mathbf{x}}.$$

12. (10 points) **(Relaxing Game)** The following puzzles consists of all Teaching Differential Geometry Teaching Assistants, please according to the lists on the right and circle all TA Names.

- Each correct TA names carries 1 point.
- The maximum score of this question is 10 points.

2024 TDG EPYMT Teaching Assistants

M	Y	I	I	O	I	M	I	A	X	A	E	M	M
A	N	W	E	K	R	M	H	J	A	A	A	W	S
A	C	C	X	H	A	N	A	K	M	R	J	E	A
K	A	L	L	H	I	I	A	A	O	L	K	V	E
Y	L	I	A	L	J	L	K	L	C	O	B	L	A
M	E	V	N	E	A	T	X	W	R	R	Y	A	N
L	X	E	A	A	C	C	Y	E	A	N	M	A	Y
K	M	L	L	H	K	J	M	T	M	N	W	V	A
Y	T	Y	M	C	Y	A	W	H	A	N	E	V	X
R	I	A	K	I	S	M	M	O	S	A	A	H	I
T	I	R	X	M	A	E	T	M	V	C	G	H	R
G	N	I	M	M	A	S	X	A	T	O	B	Y	I
T	M	K	W	M	N	A	Y	S	N	S	T	W	K
E	N	E	L	S	O	N	A	G	L	V	S	E	I

MAX
KAIKWAN
THOMAS
TOBY
MICHAEL
MARCO
RYAN
CLIVE
MING
ALEX
JAMES
JACKY
NELSON

Play this puzzle online at : https://thewordsearch.com/puzzle/7436054/

Figure 1: Teaching Assistant Name Puzzle

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